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On the equivalence of linearization and formal symmetries as integrability tests for evolution equations

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Abstract. Gürses' integrability test consists of the compatibility of the linearized equation with an eigenvalue equation and leads to the recursion operator. This test is applied to quasilinear fifth-order equations and the same classification as the 'formal symmetry' method of Mikhailov *et al* is obtained. The same classification for polynomial equations is obtained using Fokas' test, i.e. the existence of one higher-order symmetry. It is shown that the recursion operators of a specific form can be constructed using symmetries and conserved covariants and the recursion operators for polynomial equations are obtained with this method.

1. Introduction

Integrability tests for nonlinear evolution equations aim to characterize equations solvable by an inverse spectral transformation. The existence of higher-order symmetries, conserved densities, recursion operators and Hamiltonian structures are crucial characteristics of these equations. A comprehensive study of these methods is given in [1].

We recall that symmetries are solutions of the linearized equation and that recursion operators are linear operators that send symmetries to symmetries. In [1], the existence of an infinite number of symmetries (hence a recursion operator) is given as a definition for integrability. Recently two new integrability tests closely related to recursion operators have been proposed. One is the 'formal symmetry' approach of Mikhailov–Shabat–Sokolov (MSS) [2] and the other is the 'linearized eigenvalue equation' of Gürses [3].

In section 3, we give a brief review of the integrability test of Fokas [1] and Gürses [3] and we apply these tests, respectively, to polynomial and quasilinear fifth-order equations to obtain the same classification as in MSS. We show that Gürses' method is equivalent to the formal symmetry method, because the existence of a linear eigenvalue problem gives the same conserved density conditions. An advantage of this test is the use of differential operators (as opposed to pseudo-differential operators) but the choice of the orders of these operators has some subtleties. The availability of alternative methods is believed to be useful in further applications.

In section 4, we consider the direct construction of the recursion operators. We show that recursion operators that involve a single integration operation in each term (equation (4.1)) can be expressed in terms of conserved covariants and symmetries. Starting with this ansatz we obtain the recursion operators of the polynomial equations in the MSS classification in an explicit simple form. In fact, the recursion operators of these equations are known implicitly, because the recursion operator for the modified Sawada–Kotera/Kaup (equation (3.3*c*)) equation has been obtained in [4], and since all other equations are related to this via

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Miura-type transformations, their recursion operators can be constructed ([1], proposition 2.1). A recursion operator of this type has also been given by Sokolov in [5].

2. Basic definitions

We consider evolution equations of the form $u_t = F[u]$, where $u = u(x,t)$ and $F[u]$ is a differential function of u, i.e. a differentiable function of the derivatives of u with respect to x up to an arbitrary but finite order. Furthermore we assume that F is independent of x and t. We will introduce our notation below and define the concepts that will be used in this paper.

For a general review of the symmetry approach to integrable equations and some useful results we refer to [1, 2, 6].

In the following, $\mathrm{D} = \mathrm{d}/\mathrm{d}x$ and $\mathrm{D}^{-1} = \int_{-\infty}^{x}$ and it is assumed that u and all its partial derivatives vanish at $x = -\infty$. The juxtaposition of operators denotes composition, while subscripts and primes denotes componentwise differentiation of operators. We will use the term 'pseudo-differential operator' for the expression $\sum_{i=-\infty}^{n} A_i D^i$. Otherwise, the operators that involve integral operators but which are given in closed form will be called 'integro-differential operators'.

The *linearized operator* associated with the differential function $F[u]$ is denoted by F_* and defined as $F_* = \sum_{i=0}^{m} (\partial F/\partial u_i)\mathrm{D}^i$ where $u_i = (\partial^i u/\partial x^i)$ [2].

The differential polynomial $F[u]$ is said to have *fixed scaling weight s* if it transforms as $F[u] \to \lambda^s F[u]$ under the scaling $(x, u) \to (\lambda^{-1}x, \lambda^d u)$ and d is called the *weight* of u.

A differential function σ is called a *symmetry* if it satisfies the linearized equation , i.e. $\sigma_t = F_*\sigma$ [2]. A *recursion operator* is a linear operator R such that $R\sigma$ is a symmetry whenever σ is a symmetry (see [6], p 318). It can also be defined as a solution of the operator equation $R_t + [R, F_*] = 0$. Recursion operators generate flows that commute with F, if the operator has the *hereditary* property, these flows commute among each other, and symmetries are higher-order equations in the hierarchy [1].

A differential function ρ is called a *conserved density*, if there exists a differential polynomial φ such that $\rho_t = \mathrm{D}\varphi$. Similarly γ is called a *conserved covariant* if it satisfies the equation $\gamma_t + F_*^{\dagger}\gamma = 0$, where $F_*^{\dagger}$ is the adjoint of F_*. These two concepts are related as follows: the gradient of a differential function ρ is defined as $\delta\rho/\delta u = \sum_i (-1)^i \mathrm{D}^i \partial\rho/\partial u_i$, the gradients of conserved densities are conserved gradients, but not every conserved covariant is a gradient function [1].

On the space of differential functions we define an inner product by $\langle f, g\rangle = \int_{-\infty}^{\infty} (fg)\,\mathrm{d}x$. Adjoints of differential operators are defined via this inner product. If R is a recursion operator its adjoint $R^{\dagger}$ satisfies the equation $R_t^{\dagger} - [R^{\dagger}, F_*^{\dagger}] = 0$ and $R^{\dagger}$ maps conserved covariants to conserved covariants [1].

There are slight differences in the terminology used in [1, 2, 6], cited above. Mainly, in [6], symmetries are defined starting from the symmetry group of a differential equation. We give a brief review of these concepts in order to relate our definitions to the literature on the subject.

The *symmetry group* of a system of differential equations is a local group of transformations of the *independent* and *dependent variables* sending solutions to solutions (definition 2.23 in [6]). A *classical symmetry* (or Lie-point symmetry) is an infinitesimal generator v of such a group, and it is of the form

$$\sum_{k=1}^{n} \xi^k \frac{\partial}{x^k} + \sum_{\alpha=1}^{a} \phi^{\alpha} \frac{\partial}{u^{\alpha}}. \tag{2.1}$$

This transformation induces transformations of the derivatives of the dependent variables, hence we obtain a transformation on the space of the independent varibles, the dependent variables and their derivatives. The *prolongation* of $\boldsymbol{v}$ denoted by $\mathrm{pr}\,\boldsymbol{v}$, is the infinitesimal generator of this transformation, which is of the form

$$\mathrm{pr}\,\boldsymbol{v} = \sum_{k=1}^{n} \xi^k \frac{\partial}{\partial x^k} + \sum_{\alpha=1}^{a} \phi^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{J,\alpha} \phi^J \frac{\partial}{\partial u_J^\alpha} \tag{2.2}$$

where the components are given by theorem 2.36 in [6]. Generalized vector fields are vector fields of the above form, where the components are now differential functions, depending on arbitrary derivatives of the dependent variables.

It can be shown that for any generalized symmetry $\boldsymbol{v}$, one can find its *evolutionary representative* of the form

$$\boldsymbol{v}_Q = \sum_{\alpha=1}^{a} Q^\alpha \frac{\partial}{\partial u^\alpha} \tag{2.3}$$

such that $\boldsymbol{v}_Q$ is a symmetry if and only if $\boldsymbol{v}$ is, and Q is called the *characteristic* of the symmetry (proposition 5.5 in [6]). Then, from the prolongation formula (2.39) in [6], the ϕ_α^Js are total derivatives, and the action of the prolongation of $\boldsymbol{v}_Q$ on the differential equation is just the linearized equation for Q. Thus symmetries in the above sense are characteristics of infinitesimal generators.

3. Comparison of integrability tests

In section 3.1 we use Fokas' test to obtain a classification of polynomial equations with fixed weight. This method gives the polynomial equations in the classification of MSS, and an additional equation which is linearizable. In section 3.2, we apply Gürses' test to quasilinear equations and we show that it gives the same conserved density conditions as MSS, hence it leads to the same classification. We discuss the problems in the choice of the operators H and K and their relation to the recursion operator.

3.1. Classification using the existence of symmetries

In [1], Fokas considers symmetries, i.e. solutions of the linearized equation, as the basic feature of integrable equations. The existence of an infinite number of time independent, non-Lie point symmetries is given as a definition for integrability and it is conjectured that the existence of one such symmetry is sufficient for integrability. We use this criterion to classify the fifth-order *polynomial* evolution equations $u_t = F[u]$, where F has *fixed scaling weight* under the scaling $(x, u) \to (\lambda^{-1}x, \lambda^d u)$. It can be seen that nonlinearity gives a bound on the weight of u and one can only have $\mathrm{wt}(u) = 0, 1, 2$. For $\mathrm{wt}(u) = 0$, F is assumed to be independent of u. For each weight, we write down differential polynomials $F[u]$ and $\sigma[u]$ with linear terms u_5 and u_7, and we require that $\sigma[u]$ be a symmetry for the equation $u_t = [u]$. From these requirements we determine the coefficients in $\sigma[u]$ and $F[u]$.

The list of fifth-order equations admitting seventh-order symmetries are given by the equations (3.1)–(3.4). The equations obtained by the transformations $u \to u_1$ and $u \to u_1 + bu^2$ are called respectively the *potential* and *modified* form of the original one and they are denoted by the prefixes p and m. The basic equations in the classification are the KdV, Sawada–Kotera (SK) and Kaup (K) equations. Equation (3.1a) denoted by (F) is linearizable via the transformation $u = \frac{5}{\beta} \ln v$ hence both are considered to be integrable [2, 7]. We note that the canonical density ρ_4 of MSS is not conserved for these equations but that their list excludes equations linearizable via point transformations.

(i) *Linearizable equations:*

$$\text{(pF)}\ u_t = u_5 + \beta(u_4u_1 + 2u_3u_2) + \beta^2(\tfrac{2}{5}u_3u_1^2 + \tfrac{3}{5}u_2^2u_1) + \tfrac{2}{25}\beta^3u_2u_1^3 + \tfrac{1}{625}\beta^4u_1^5 \qquad (3.1a)$$

$$\text{(F)}\ u_t = u_5 + \beta(u_4u + 3u_3u_1 + 2u_2^2) + \beta^2(\tfrac{2}{5}u_3u^2 + 2u_2u_1u + \tfrac{3}{5}u_1^3) + \beta^3(\tfrac{2}{25}u_2u^3 + \tfrac{6}{25}u_1^2u^2) + \tfrac{1}{125}\beta^4u_1u^4. \qquad (3.1b)$$

(ii) Nonlinear fifth-order equations admitting a seventh-order symmetry:

$\mathrm{wt}(u) = 0$:

$$\text{(pmKdV)}\ u_t = u_5 + \beta(u_3u_1^2 + u_2^2u_1) + \tfrac{3}{50}\beta^2u_1^5 \qquad (3.2a)$$

$$\text{(pmSK–K)}\ u_t = u_5 + \beta u_3u_2 - \tfrac{1}{5}\beta^2(u_1^2u_3 + u_2^2u_1) + \tfrac{1}{625}\beta^4u_1^5 \qquad (3.2b)$$

$\mathrm{wt}(u) = 1$:

$$\text{(mKdV)}\ u_t = u_5 + \beta(u_3u^2 + 4u_2u_1u + u_1^3) + \beta^2\tfrac{3}{10}u_1u^4 \qquad (3.3a)$$

$$\text{(pKdV)}\ u_t = u_5 + \beta(u_3u_1 + \tfrac{1}{2}u_2^2) + \beta^2\tfrac{1}{10}u_1^3 \qquad (3.3b)$$

$$\text{(mSK–K)}\ u_t = u_5 + \beta(u_3u_1 + u_2^2) - \beta^2(\tfrac{1}{5}u_3u^2 + \tfrac{4}{5}u_2u_1u + \tfrac{1}{5}u_1^3) + \beta^4\tfrac{1}{125}u_1u^4 \qquad (3.3c)$$

$$\text{(pSK)}\ u_t = u_5 + \beta u_3u_1 + \beta^2\tfrac{1}{15}u_1^3 \qquad (3.3d)$$

$$\text{(pK)}\ u_t = u_5 + \beta(\tfrac{4}{3}u_3u_1 + u_2^2) + \beta^2\tfrac{16}{135}u_1^3 \qquad (3.3e)$$

$\mathrm{wt}(u) = 2$:

$$\text{(KdV)}\ u_t = u_5 + \beta(u_3u + 2u_2u_1) + \tfrac{3}{10}\beta^2u_1u^2 \qquad (3.4a)$$

$$\text{(SK)}\ u_t = u_5 + \beta(u_3u + u_2u_1) + \tfrac{1}{5}\beta^2u_1u^2 \qquad (3.4b)$$

$$\text{(K)}\ u_t = u_5 + \beta(u_3u + \tfrac{5}{2}u_2u_1) + \tfrac{1}{5}\beta^2u_1u^2. \qquad (3.4c)$$

In section 4 we will obtain recursion operators for equations (3.2)–(3.4).

3.2. Integrability test using linearization

In this section we consider applications of the integrability test proposed in [3]. We will show that this test with a linear eigenvalue problem is equivalent to the construction of a recursion operator in terms of differential operators and applied to fifth-order quasilinear equations gives the same classification as MSS. We briefly recall the following:

(i) Gürses' integrabilty test: the equation $u_t = F[u]$ is integrable if there exists an operator $L = \sum_{i=0}^{k}\lambda^iL^{(i)}$, $\mathrm{Ord}L^{(i)} < \mathrm{Ord}L^{(0)}$, such that the eigenvalue equation $L\sigma = 0$ is compatible with the linearized equation $\sigma_t = F_*\sigma$. It can be seen that this compatibility condition is equivalent to the equation $(L_t + [L, F_*])\sigma = 0$. These equations are to hold identically in λ and $\mathrm{D}^k\sigma$ for $k < \mathrm{Ord}L^{(0)}$.

(ii) MSS integrability test: the equation $u_t = F[u]$ is integable if there exists a pseudo-differential operator $\tilde{L}$ such that the operator equation $\tilde{L}_t + [\tilde{L}, F_*] = 0$ holds up to sufficiently low orders. The operator $\tilde{L}$ is called a formal symmetry.

Remark 3.1. Recall that an integro-differential operator R is a recursion operator for the equation $u_t = F$ iff $R_t + [R, F_*] = 0$. It is clear that if a recursion operator has a finite number of integro-differential terms, the coefficients in its expansion in powers of D^{-1} will be determined from the first few terms. Thus in such cases a formal symmetry is just the expansion of a recursion operator.

We will show that under certain general assumptions the existence of an eigenvalue problem is equivalent to the existence of a recursion operator.

Consider an eigenvalue problem, linear in λ, i.e. let $L = H - \lambda K$ and consider the compatibility equation

$$(H_t - \lambda K_t + [H, F_*] - \lambda[K, F_*])\sigma = 0 \tag{3.5}$$

for all σ such that $\sigma_t = F_*\sigma$ and $H\sigma - \lambda K\sigma = 0$. By premultiplying this with K^{-1} and making use of the eigenvalue equation we find that

$$((K^{-1}H)_t + [(K^{-1}H), F_*])\sigma = 0 \tag{3.6}$$

for all eigenfunctions of L. Thus, provided that the eigenfunctions are dense in the symmetries $K^{-1}H$ is a recursion operator. Conversely, assume that the equation admits a recursion operator R. Let σ_0 be a symmetry, i.e. some differential function which is a solution of the linearized equation. Let $\sigma_n = R^n\sigma_0$ and define $\sigma = \sum_{n=-\infty}^{\infty} \lambda^{-n}\sigma_n$ as a formal series. Then the equation $R\sigma = \lambda\sigma$ holds identically in λ, which is an eigenvalue problem with an integro-differential operator. Assume, furthermore, that there is a differential operator K such that $H = KR$ is a differential operator (see remark 3.2). Multiplying $R\sigma = \lambda\sigma$ with K from the left we obtain the compatibility of the eigenvalue problem with the linearized equation. Thus *the existence of an eigenvalue problem is equivalent to the existence of a recursion operator, provided that the eigenfunctions are dense in the symmetries and the recursion operator is factorizable in terms of differential operators.*

Remark 3.2. Assume that R is of the form $R = R_0 + \sum_{k=1}^{N} \phi_k \mathrm{D}^{-1}\psi_k$, where R_0 is a differential operator and ϕ_ks and ψ_ks are differential functions. Then it is always possible to find a differential operator K of order N such that $K\phi_k = 0$ for $k = 1, \ldots, N$, hence KH is a differential operator. In the next section we will show that the recursion operators for equations (3.2)–(3.4) are of this form.

We applied Gürses' test to fifth-order quasilinear equations of the form $u_t(x, t) = F[u] = u_5 + f(u, u_1, u_2, u_3)$. The classification of these equations have been obtained in [2], thus our aim is to compare the two methods as far as applicability is concerned. We shall describe here the computations in detail and discuss the pitfalls in the applications. The existence of the operators H and K depends on the existence of certain conserved densities, as these are found to be identical to the canonical densities of MSS they are not presented here. These conserved density conditions are found at a fairly early stage in the computation of the operators H and K, hence the classification problem can be solved with a reasonable amount of computation. The complete determination of the operators H and K is straightforward but cumbersome. They have been obtained only for the polynomial equations in the classification but the results are not presented. The recursion operators for these equations are obtained directly. The computations are done with REDUCE on a PC with a 486 microprocessor, using an integration package for differential polynomials [8]. We now describe the solution process.

In the application of Gürses' test it is very important to start with H and K with appropriate orders. Assuming that R is of the form given in remark 3.2, the order of K is N. Then the Ord(H)=Ord(K)+Ord(R). Thus as a starting point for the classification we need to know the form of the recursion operators to be looked for. For the well known KdV hierarchy Ord$(R) = 2$ and $N = 1$. In section 4, we will obtain recursion operators for other integrable polynomial equations which are of order 6 with $N = 2$. It turns out that choosing H and K as differential operators of orders respectively 8 and 2 we obtain the conserved density conditions of MSS, hence a complete classification. However if the orders of H and K differ by 2 the Sawada–Kotera and Kaup hierarchies are missed.

Once the orders of the operators are determined, the operator equation $((H - \lambda K)_t + [(H - \lambda K), F_*])\sigma = 0$ can be solved interactively. More specifically we parametrize H and K as

$$\begin{aligned} H &= \mathrm{D}^8 + h_7\mathrm{D}^7 + h_6\mathrm{D}^6 + h_5\mathrm{D}^5 + h_4\mathrm{D}^4 + h_3\mathrm{D}^2 + h_2\mathrm{D}^1 + h_0 \\ K &= \mathrm{D}^2 + k_1\mathrm{D} + k_0. \end{aligned} \tag{3.7}$$

The compatibility equation is a first-order equation in λ. The coefficeint of λ^1 and λ^0 gives 14 equations which are the coefficients of $\mathrm{D}^k\sigma$, $k = 7, \ldots, 0$. These equations lead to first-order equations solvable by quadratures. Finally we obtain k_1, k_0, and h_i for $i = 0, \ldots, 5$ in terms of h_6 and h_7 and the evolution equations $(h_6)_t = f[h_6, h_7, u]$ and $(h_7)_t = g[h_7, h_7, u]$. It remains to solve 3 more equations which are the coefficients of $\lambda^0\mathrm{D}^k\sigma$ for $k = 0, 1, 2, 3$. These first two of there equations are relatively simple and, after integration, lead to linear equations for h_6 and h_7. The coefficients of $\lambda^0\mathrm{D}^1\sigma$ and $\lambda^0\sigma$ are nonlinear expressions of the derivatives of h_6 and h_7. The nonlinear parts are eliminated interactively and they also lead to linear algebraic equations after integration. These integrations give two additional conserved densities. At this stage, it was not possible to check the compatibility of this linear system in its generality due to computer limitations. However, as the conserved density conditions (the canonical densities of MSS) are sufficient to give a classification the problem is solved in this respect. After this point due to computer limitations, the remaining equations are solved only for the polynomial equations in the classification, and the operators K and H are obtained.

We note that both tests have a 'trial and error' component: in the MSS test the order of the formal symmetry and, in Gürses' test, the orders of K and H are not specified. In the applications, these parameters should be varied until the resulting conserved density conditions give a complete classification.

4. Construction of the recursion operators

The integrability tests considered above lead to a recursion operator but its construction with the above method is difficult in practice. Here we show that if the recursion operator is of a specific form (4.1), it is determined in terms of symmetries and conserved densities, and we show that the recursion operators for polynomial equations in the MSS classification are all of this form.

The knowledge of the first few symmetries and conserved covariants provide a clue for the degree of the recursion operator, because the recursion operator R sends symmetries to symmetries and its adjoint $R^\dagger$ sends conserved covariants to conserved covariants. For example the KdV and mKdV hierarchies have symmetries at every odd order, hence the recursion operator of least order that sends local symmetries to local symmetries is expected to be of order 2. On the other hand the Sawada–Kotera and Kaup equations arising from a third-order spectral problem [9] have symmetries at every odd order that is not a multiple of 3. Thus such a recursion operator of least order will be of order 6. A similar statement is true for the modified equation of these hierarchies, since they still arise from a third-order spectral problem [4]. Thus the order of the recursion operator can be guessed fairly easily.

The parametrization of the part that depends on integral operators is more subtle. Assuming a specific form for the recursion operator, namely that there is only single integration operation in each term, we have the following proposition.

Proposition 4.1. Let R be a recursion operator of the form

$$R = R_0 + \sum_{j=1}^{N} \varphi_j \mathrm{D}^{-1} \psi_j \tag{4.1}$$

where R_0 is a differential operator, and φ_j and ψ_j are independent differential functions. Then the φ_js and ψ_js are, respectively, symmetries and conserved covariants, i.e.

$$\frac{\mathrm{d}\varphi_j}{\mathrm{d}t} = F_* \varphi_j \qquad \frac{\mathrm{d}\psi_j}{\mathrm{d}t} = -F_*^\dagger \psi_j. \tag{4.2}$$

Proof. Computing the operator equation $R_t + [R, F_*] = 0$ and making use of the binomial expansion for the derivative, and of the identity $\mathrm{D}^{-1}\alpha\mathrm{D}^k = \sum_{i=1}^{k}(-1)^{i+1}(\mathrm{D}^{i-1}\alpha)\mathrm{D}^{-i+k} + (-1)^k\mathrm{D}^{-1}(\mathrm{D}^k\alpha)$, where α is any scalar function, we obtain

$$\begin{aligned} 0 = (R_0)_t + [R_0, F_*] + R_1 - R_2 + \sum_{j=1}^{N}\left[(\varphi_j)_t - \sum_{k=0}^{m} F_k(\mathrm{D}^k\varphi_j)\right]\mathrm{D}^{-1}\psi_j \\ + \sum_{j=1}^{N}\varphi_j\mathrm{D}^{-1}\left[(\psi_j)_t + \sum_{k=0}^{m}(-1)^k\mathrm{D}^k(\psi_j F_k)\right] \end{aligned} \tag{4.3}$$

where the differential operators R_1 and R_2 are given by

$$R_1 = \sum_{j=1}^{N}\sum_{k=1}^{m}\varphi_j\sum_{i=1}^{k}(-1)^{i+1}(D^{i-1}(\psi_j F_k))\mathrm{D}^{-i+k} \tag{4.4a}$$

$$R_2 = \sum_{j=1}^{N}\sum_{k=1}^{m}F_k\sum_{i=0}^{k-1}\binom{k}{i}(\mathrm{D}^i\varphi_j)\mathrm{D}^{k-i-1}\psi_j. \tag{4.4b}$$

Thus provided that the φ_js and ψ_js are independent, they shoud satisfy $(\varphi_j)_t = F_*\varphi_j$ and $(\psi_j)_t = -F_*^\dagger\psi_j$. □

Thus if the φ_js and ψ_js are, respectively, symmetries and conserved covariants, then the equation $R_t + [R, F_*] = 0$ reduces to a differential operator equation, which can be easily computed with a symbolic programming language.

The form of the recursion operator is somewhat easier to guess if the evolution equation is differential polynomial and has fixed weight under some scaling $(x, u) \to (\lambda^{-1}x, \lambda^d u)$, i.e. if $F[u] \to \lambda^s F[u]$ under such a transformation: in such cases the symmetries and conserved covariants are also differential polynomials of fixed weight, and the integral part of the recursion operator can be constructed, once its order is determined.

Example 4.2. The recursion operators for the KdV ($u_t = u_3 + 6uu_1$) and mKdV ($u_t = u_3+6u^2u_1$) equations are respectively $R = \mathrm{D}^2+4u+2u_1\mathrm{D}^{-1}$ and $R = \mathrm{D}^2+4u^2+2u_1\mathrm{D}^{-1}u$. These operators are of the form (4.1) with Ord$R = 2$, $N = 1$, $\phi_1 = u_1$ (which is a symmetry for any equation $u_t = F[u]$ where $F[u]$ is independent of x), and ψ_1 being respectively 1 and u. Note that in each case $\mathrm{wt}(\varphi_i) + \mathrm{wt}(\psi_i) = 3$.

Except for the KdV hierarchy, all equation in (3.2)–(3.4) are related to the mSKK equation (3.3c) by Miura transformations. Thus the recursion operators of all these equations should be of order 6. The first few symetries and conserved covariants for each equation are computed and the form the recursion operator is determined by the requirement that $\mathrm{wt}(\phi_j)$+$\mathrm{wt}(\psi_j) = 7$. In all cases there are two terms with integral operators, hence the form of the recursion operator is

$$R = \mathrm{D}^6 + r_5\mathrm{D}^5 + r_4\mathrm{D}^4 + r_3\mathrm{D}^2 + r_2\mathrm{D}^2 + r_1\mathrm{D} + r_0 + b_1\sigma_1\mathrm{D}^{-1}\psi_1 + b_2\sigma_2\mathrm{D}^{-1}\psi_2. \qquad (4.5)$$

The equation (4.2) is solved with REDUCE and the results are below. In each equation $F[u]$ stands for the differential polynomial in the right-hand side of the corresponding equation in (3.2)–(3.4).

$$\begin{aligned}(\mathrm{pmSKK})\ R = {} & \mathrm{D}^6+(6u_2-6u_1^2)\mathrm{D}^4+(9u_3-18u_2u_1)\mathrm{D}^3+(5u_4-22u_3u_1-13u_2^2-6u_2u_1^2+9u_1^4)\mathrm{D}^2\\ & + (u_5 - 8u_4u_1 - 15u_3u_2 - 3u_3u_1^2 - 6u_2^2u_1 + 18u_2u_1^3)\mathrm{D}\\ & + (-4u_5u_1 - 20u_3u_2u_1 + 20u_3u_1^3 + 20u_2^2u_1^2 - 4u_1^6) + 2F[u]\mathrm{D}^{-1}u_2\\ & + 2u_1\mathrm{D}^{-1}(u_6 + 5u_2u_4 - 5u_4u_1^2 + 5u_3^2 - 20u_3u_2u_1 - 5u_2^3 + 5u_2u_1^4) \qquad (4.6a)\end{aligned}$$

$$\begin{aligned}(\mathrm{mSKK})\ R = {} & \mathrm{D}^6+(6u_1-6u^2)\mathrm{D}^4+(15u_2-30u_1u)\mathrm{D}^3+(14u_3-40u_2u-31u_1^2-6u_1u^2+9u^4)\mathrm{D}^2\\ & +(6u_4-30u_3u-63u_2u_1-9u_2u^2-18u_1^2u+54u_1u^3)\mathrm{D}+(u_5-12u_4u-23u_3u_1^3\\ & - 3u_3u^2 - 15u_2^2 - 38u_2u_1u + 38u_2u^3 - 6u_1^3 + 74u_1^2u^2 - 4u^6)\\ & - 2F[u]\mathrm{D}^{-1}u - 2u_1\mathrm{D}^{-1}(u_4 + 5u_2u_1 - 5u_2u^2 - 5u_1^2u + u^5) \qquad (4.6b)\end{aligned}$$

$$\begin{aligned}(\mathrm{pSK})\ R = {} & D^6+6u_1\mathrm{D}^4+3u_2D^3+(8u_3+9u_1^2)\mathrm{D}^2+(2u_4+3u_2u_1)\mathrm{D}+(3u_5+13u_3u_1+3u_2^2+4u_1^3)\\ & - 2\mathrm{D}^{-1}(u_6 + 3u_1u_4 + 6u_3u_2 + 2u_2u_1^2) - 2u_1\mathrm{D}^{-1}(u_4 + u_2u_1) \qquad (4.6c)\end{aligned}$$

$$\begin{aligned}(\mathrm{pK})\ R = {} & \mathrm{D}^6 + 6u_1\mathrm{D}^4 + 12u_2\mathrm{D}^3 + (\tfrac{25}{2}u_3 + 9u_1^2)\mathrm{D}^2 + (5u_4 + 12u_2u_1)\mathrm{D}\\ & + (\tfrac{3}{2}u_5 + \tfrac{17}{2}u_3u_1 + \tfrac{21}{4}u_2^2 + 4u_1^3) - \tfrac{1}{2}\mathrm{D}^{-1}(u_6 + 6u_1u_4 + 12u_3u_2 + 8u_2u_1^2)\\ & - \tfrac{1}{2}\mathrm{D}^{-1}(u_4 + 4u_2u_1) \qquad (4.6d)\end{aligned}$$

$$\begin{aligned}(\mathrm{SK})\ R = {} & \mathrm{D}^6 + 6u\mathrm{D}^4 + 9u_1\mathrm{D}^3 + (11u_2 + 9u^2)\mathrm{D}^2 + (10u_3 + 21u_1u)\mathrm{D}\\ & + (5u_4 + 16uu_2 + 6u_1^2 + 4u^3) + F[u]\mathrm{D}^{-1} + 2u_1\mathrm{D}^{-1}(u_2 + \tfrac{1}{2}u^2) \qquad (4.6e)\end{aligned}$$

$$\begin{aligned}(\mathrm{K})\ R = {} & \mathrm{D}^6 + 6u\mathrm{D}^4 + 18u_1\mathrm{D}^3 + (\tfrac{49}{2}u_2 + 9u^2)\mathrm{D}^2 + (\tfrac{35}{2}u_3 + 30u_1u)\mathrm{D}\\ & + (\tfrac{13}{2}u_4 + \tfrac{41}{2}uu_2 + \tfrac{69}{4}u_1^2 + 4u^3) + F[u]\mathrm{D}^{-1} + \tfrac{1}{2}u_1\mathrm{D}^{-1}(u_2 + 2u^2). \qquad (4.6f)\end{aligned}$$

Remark 4.5. As conserved covariants are generated by repeated applications of $R^\dagger$ to a 'starting conserved covariant' say γ_0, they are more basic than conserved densities for integrability. We note that for certain equations the canonical densities of the MSS classification are all trivial.

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